AN ITERATIVE-DECOMPOSED PIECEWISE-LINEAR REPRESENTATION

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ABSTRACT

In this paper a new model description for the representation of one-dimensional piecewise-linear (PWL) characteristics is presented. The description can be efficiently applied to model both, uni-valued and multi-valued PWL characteristics, and it has the advantage of obtaining the model parameters directly from the graphic coordinates. The description is denoted as an Iterative and Decomposed (ID) model. It is described as a decomposed one, because the independent and dependent variables in the mathematical formulation, that describes an arbitrary PWL characteristic, are treated separately. It is also called iterative, because the particular representation of each PWL segment depends only on the value of one parameter included in the mathematical formulation. An example is presented and it illustrates the potential application of this description in the modeling and analysis of nonlinear circuits.

1. INTRODUCTION

A model description is a mathematical representation that defines the specific system functionality. In PWL theory, it is common to describe the functionality of nonlinear systems by PWL characteristics. In systems like PWL circuits, those characteristics must be mathematically modeled (described) in order to make possible the circuit analysis. Much effort has been put in a search for a particular model description in which all kinds of PWL curves (uni-valued and multi-valued) can be modeled [1],[2],[3]. An approach to overcome this problem has been the Generalized-Linear-Complementary-Problem (GLCP) [4],[5] and the Decomposed-Parametric (DP) [6] models. The GLCP model can be applied on the representation of all kind of PWL functions, but it has the disadvantage of producing too long mathematical expressions. Although the DP model produces more simple expressions than the GLCP model, according with [7], it is only limited to handle de so-called four attainable types of PWL characteristics.

The ID model which is proposed in this paper can be efficiently applied to describe all kind of one-dimensional PWL characteristics (uni-valued and multi-valued) and its mathematical formulation is not so heavy to be computed because it is based on an iterative system of linear equations. The model is denoted as decomposed, because the independent variable x and the dependent variable $y = f(x)$ are included separately in a system of linear equations. The model is also denoted as iterative, because there is a parameter k, which selects a specific segment from the set of L segments that constitutes the complete PWL representation.

2. THE ITERATIVE-DECOMPOSED MODEL

Let the PWL characteristic depicted in Figure 1 be characterized by L segments and $L+1$ coordinates: $\{(X_1,Y_1),(X_2,Y_2),\ldots,(X_L,Y_L),(X_{L+1},Y_{L+1})\}$.

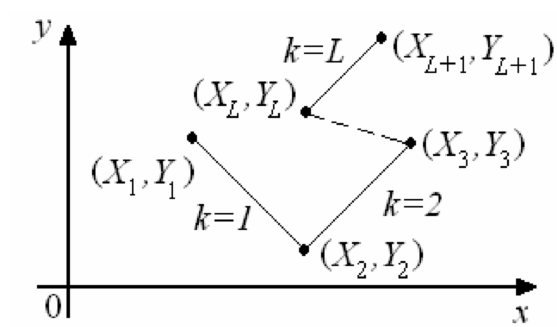


Figure 1. PWL characteristic and its coordinates.

An ID formulation for this characteristic can be obtained as follows.

Firstly, the particular problem of obtaining the line equation for any of the *k*-th segments included in the PWL characteristic is analyzed. The graphic exposition of this problem is depicted in Figure 2.

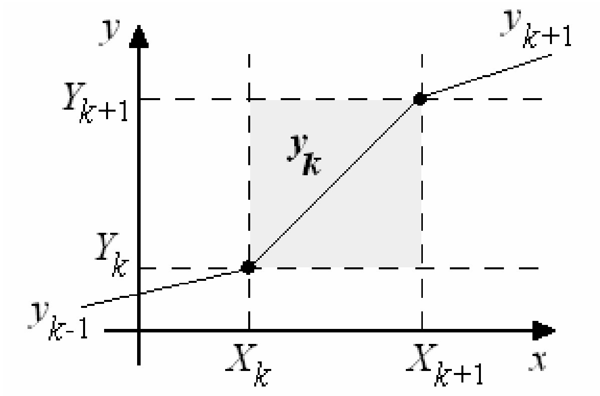


Figure 2. A k-th segment belonging to a PWL curve.

The line equation for the *k*-th segment is given as:

$$y_k = \left(\frac{Y_{k+1} - Y_k}{X_{k+1} - X_k}\right) x + \left(\frac{Y_k X_{k+1} - Y_{k+1} X_k}{X_{k+1} - X_k}\right) \tag{1}$$

then, (1) is factored into the form

$$y_k = P_k Y_k + Q_k Y_{k+1} \tag{2}$$

Where P_k and Q_k are variables defined by the determinant relations:

$$P_k = \frac{\begin{vmatrix} x & X_{k+1} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} X_k & X_{k+1} \\ 1 & 1 \end{vmatrix}} \;,\; Q_k = \frac{\begin{vmatrix} X_{k+1} & x \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} X_k & X_{k+1} \\ 1 & 1 \end{vmatrix}} \tag{3}$$

the relation between (2) and (3) is expressed as the system of linear equations given in:

$$\begin{bmatrix} y_k \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} Y_k & Y_{k+1} \\ X_k & X_{k+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_k \\ Q_k \end{bmatrix} \tag{4}$$

the above linear system represents the *k*-th line equation included in the PWL characteristic to be modeled.
Finally, a complete mathematical expression which assures a description for the PWL characteristic depicted in Figure 1 is obtained if (4) is generalized for all *L* segments. Thus, the ID-model description can be written:

$$\begin{bmatrix} y \\ x \\ 1 \end{bmatrix} = \sum_{i=1}^{L} \begin{bmatrix} Y_i & Y_{i+1} \\ X_i & X_{i+1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_i \\ Q_i \end{bmatrix} C_i(k) \tag{5}$$

with the condition:

$$P_i(x), Q_i(x), P_i(y), Q_i(y) \geq 0 \tag{6}$$

where $C_i(k)$ is denoted as an activation coefficient and it is defined as follows:

$$C_i(k) = \frac{1}{(i)!(L-i)!} \prod_{m=0}^{i-1} |k - m| \prod_{n=1}^{L-i} |k - (i+n)| \tag{7}$$

for $i = 1 \ldots L-1$, and

$$C_i(k) = \frac{1}{(L)!} \prod_{j=0}^{L-1} |k - j| \tag{8}$$

for $i = L$.

the variables P_i and Q_i in (5) control the interval where a *k*-th line equation does exists. This is expressed in the condition (6) and it is demonstrated as follows.
Firsty, the notations ΔX_i and ΔY_i are defined as:

$$\Delta X_i = X_{i+1} - X_i \;,\; \Delta Y_i = Y_{i+1} - Y_i \tag{9}$$

then, mathematical expressions, for the variables P_i and Q_i, can be obtained if the following decomposed linear equations, which are included in (5), are algebraically manipulated.

$$y = Y_i P_i + Y_{i+1} Q_i \tag{10}$$

$$x = X_i P_i + X_{i+1} Q_i \tag{11}$$

$$1 = P_i + Q_i \tag{12}$$

After solving for P_i in (12) and substituting it into (10) and (11), Q_i can be expressed as

$$Q_i(x) = \frac{x}{\Delta X_i} - \frac{X_i}{\Delta X_i}, Q_i(y) = \frac{y}{\Delta Y_i} - \frac{Y_i}{\Delta Y_i}, \tag{13}$$

A similar analysis when solving for Q_i in (12) and then it is substituted into (10) and (11) yields:

$$P_i(x) = \frac{-x}{\Delta X_i} + \frac{X_{i+1}}{\Delta X_i}, P_i(y) = \frac{-y}{\Delta Y_i} + \frac{Y_{i+1}}{\Delta Y_i}, \tag{14}$$

If the equations (13) and (14) are subjected to the expressed condition in (6) results:

$$Q_i(x) = (x - X_i)/\Delta X_i \geq 0, \qquad x \geq X_i \tag{15}$$

$$Q_i(y) = (y - Y_i)/\Delta Y_i \geq 0, \qquad y \geq Y_i \tag{16}$$

$$P_i(x) = (X_{i+1} - x)/\Delta X_i \geq 0, \qquad x \leq X_{i+1} \tag{17}$$

$$P_i(x) = (Y_{i+1} - y)/\Delta Y_i \geq 0, \qquad y \geq Y_{i+1} \tag{18}$$

The equations (15),(16),(17),(18) indicate the intervals where a linear segment belonging to a PWL characteristic exists. The shadow area depicted in Figure 2 results by the intersection of these intervals for the *k*-th PWL segment.

3. EQUIVALENT CIRCUIT

The *i*-th linear system in (5) can be electrically modeled by the circuit shown in Figure 3.

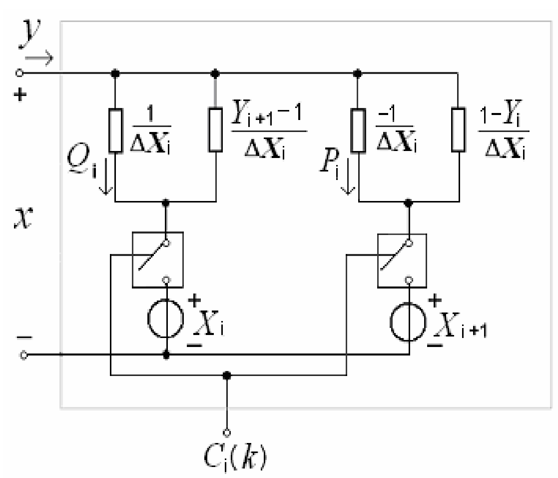


Figure 3. Equivalent circuit for a PWL segment.

In this circuit, the variables Q_i and P_i, have a physical meaning because they represent the current flowing through the conductances whose values are ΔX_i^{-1} and $-\Delta X_i^{-1}$ respectively.

The electrical diagram includes two switches which are controlled simultaneously by $C_i(k)$. Figure 4 shows the symbol used to represent them. On the one hand, it can be seen that when the switches are open, these variables are equal to zero, it occurs when $C_i(k) = 0$, on the other hand, when $C_i(k) = 1$, the switches are closed and the variables (currents Q_i and P_i) are different to zero.

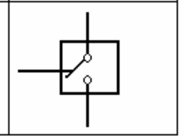
Element	Symbol
Switch controlled by $C_i(k)$	

Figure 4. Switch symbol included in the equivalent circuit.

The *L* linear systems included in (5) can be represented by the equivalent circuit shown in Figure 5. Each block is a simplified representation of the circuit shown in Figure 3.

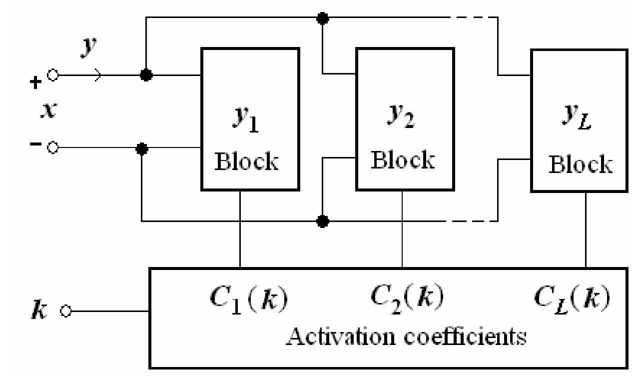
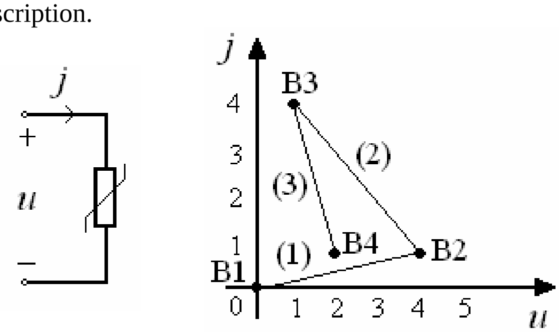


Figure 5. Equivalent circuit for the ID-model.

4. EXAMPLE

Consider the theoretical nonlinear element and its PWL characteristic depicted in Figure 6. Determine its ID model description.



(a) (b)

Figure 6.(a) Nonlinear element, (b) PWL characteristic for the nonlinear element.

An analysis from Figure 6(b) shows the existence of three equations labeled as (1), (2) and (3). The four graphic coordinates are designed as B1, B2, B3, B4, and they are summarized in Table 1.

Table 1. Graphic co-ordinates for the piecewise-linear characteristic of Figure.6(b).

i	Coordinate	u_i	j_i
1	B1	0	0
2	B2	4	1
3	B3	1	4
4	B4	2	1

There are three segments, therefore *L*=3 and *k*=1,2,3. Substituting *L* and the co-ordinates of Table 1 into (5), results:

$$\begin{bmatrix} j \\ u \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} C_1(k) + \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ Q_2 \end{bmatrix} C_2(k) + \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_3 \\ Q_3 \end{bmatrix} C_3(k)$$

Substituting *L* into (7) and (8), the activation coefficients $C_i(k)$ are given by:

$$C_1(k) = \frac{k|k-2||k-3|}{2},$$
$$C_2(k) = \frac{k|k-1||k-3|}{2},$$
$$C_3(k) = \frac{k|k-1||k-2|}{6}$$

When *k*=1, $C_1(k)$=1, $C_2(k)$=0, $C_3(k)$=0, and the resulting linear system is:

$$\begin{bmatrix} j \\ u \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix}$$

The line equation is $j = (1/4)u$ limited by the intervals *u*=[0,4] and *j*=[0,1].

When *k*=2, $C_1(k)$=0, $C_2(k)$=1, $C_3(k)$=0, and the resulting linear system is:

$$\begin{bmatrix} j \\ u \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}$$

The line equation is $j = -u + 5$ limited by the intervals *u*=[1,4] and *j*=[1,4].

When *k*=3, $C_1(k)$=0, $C_2(k)$=0, $C_3(k)$=1 and the resulting linear system is:

$$\begin{bmatrix} j \\ u \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_3 \\ Q_3 \end{bmatrix}$$

The line equation is $j = -3u + 7$ limited by the intervals *u*=[1,2] and *j*=[1,4].

Figure 7 shows the equivalent circuit which models the PWL characteristic analyzed in this example.

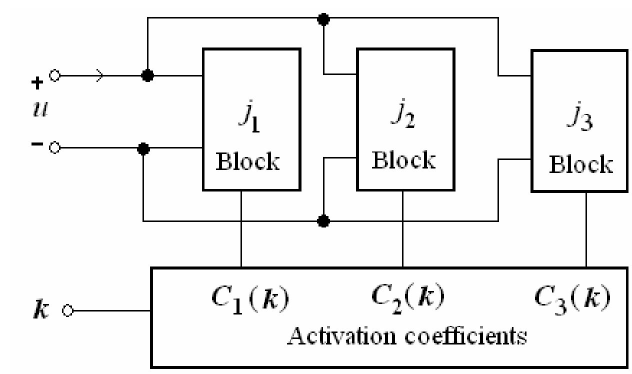


Figure 7.Electrical equivalent circuit for the example.

5. CONCLUSION

The proposed ID model can describe any arbitrary PWL function (uni-valued or multi-valued) by considering as model parameters its graphic coordinates. The interval which bounds the ranges where a linear segment is valid, is determined by the variables P_i and Q_i. These variables have a physical meaning in the electrical model. Finally, it is important to point that the electrical model has simulation capability because it is compatible with iterative simulation techniques.

6. REFERENCES

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